

Einstein Gravity from Conformal Gravity

Juan Maldacena

*School of Natural Sciences, Institute for Advanced Study
Princeton, NJ 08540, USA*

We show that that four dimensional conformal gravity plus a simple Neumann boundary condition can be used to get the semiclassical (or tree level) wavefunction of the universe of four dimensional asymptotically de-Sitter or Euclidean anti-de Sitter spacetimes. This simple Neumann boundary condition selects the Einstein solution out of the more numerous solutions of conformal gravity. It thus removes the ghosts of conformal gravity from this computation.

In the case of a five dimensional pure gravity theory with a positive cosmological constant we show that the late time superhorizon tree level probability measure, $|\Psi[g]|^2$, for its four dimensional spatial slices is given by the action of Euclidean four dimensional conformal gravity.

1. Introduction

Conformal gravity is an intriguing theory of gravity. It is a theory of gravity in four dimensions with an action given by the square of the Weyl tensor, $S_{\text{conf}} = \int d^4x \sqrt{g} W^2$. It is a theory sensitive to angles, but not distances. In fact, a Weyl transformation of the metric, $g_{\mu\nu} \rightarrow \Omega^2(x)g_{\mu\nu}$, is an exact symmetry of this action. It has appeared periodically in the literature for various reasons. It was considered as a possible UV completion of gravity [1,2,3], and references therein. It was also useful for constructing supergravity theories, see e.g. [4]. It has recently emerged from the twistor string theory [5]. It has also appeared as a counter term in AdS_5 or CFT_4 computations [6,7]. It was seldom taken seriously because it has ghosts, due to the fact that the equations of motion are fourth order¹.

Here we would like to point out a couple of interesting connections between conformal gravity and ordinary gravity in AdS or dS . We show that conformal gravity with certain future boundary conditions is equivalent to ordinary gravity in asymptotically de-Sitter space. Alternatively, we can say that by setting the ghost fields to zero in the future of de-Sitter, we get a wavefunctional for the metric which is the same as the one given by Bunch Davies (or Hartle Hawking) at tree level. A similar relation is present for Euclidean spaces which are asymptotically $EAdS$, or hyperbolic space.

It was observed in [10] that the renormalized on shell action of four dimensional Einstein gravity in asymptotically hyperbolic Einstein spaces is given by the the action of conformal gravity. As usual, this action is evaluated on a solution of Einstein gravity. It is a well known fact, that the solutions of Einstein gravity are also solutions of conformal gravity. But conformal gravity has other solutions. If we were able to select, in a simple way, the solutions of Einstein gravity from the solutions of conformal gravity, then we can forget about the Einstein action and use instead the conformal gravity action in the bulk. Actually, it is very easy to select the solutions of Einstein gravity. We simply need to impose a Neumann boundary condition on the metric at the boundary. Then this simple boundary condition is eliminating the ghosts and rendering the theory equivalent to ordinary pure Einstein gravity with a cosmological constant.

The discussion in this paper is purely classical, or tree level, but it is non-linear. And it would be interesting if one could somehow use this to construct a full quantum theory

¹ See [8] for ideas on how to deal with ghosts and [9] for some criticisms of that idea.

based on conformal gravity alone, since an $\mathcal{N} = 4$ supersymmetric version of conformal gravity is possibly finite, see [11] for a review.

It was observed in [6,12] that the action of four dimensional conformal gravity appears as a logarithmically divergent counterterm in computations in five dimensional asymptotically Hyperbolic spaces. It appears as the coefficient of the holographic Weyl anomaly. An interesting situation arises when we consider the de-Sitter version of this computation. In de-Sitter space, one is often interested in computing the probability measure for the spatial slices at superhorizon distances $|\Psi[g]|^2$. This is the tree level solution of the (super-horizon) measure problem in such a universe. It turns out that this probability measure is given by the action of conformal gravity in four dimensions $|\Psi[g]|^2 = e^{-S_{\text{conf}}[g]}$. This is due to two facts. First the fact noted in [13,14,15] that the de-Sitter wavefunction can be computed by a certain analytic continuation from the Euclidean AdS one. The only term that becomes real is a term that comes from analytically continuing the logarithmic divergence. This produces a finite term given by the action of conformal gravity.

This paper is organized as follows. In section two we discuss a conformally coupled scalar field with a fourth derivative action. This serves as a toy example for conformal gravity. In section three we discuss the relation between conformal gravity with a boundary condition and ordinary Einstein gravity in AdS or dS . In section four we consider the black hole contributions to the partition function of conformal gravity. In section five we make a side comment regarding the Hartle Hawking measure factor and the 3-sphere partition function of a possible dual boundary CFT . In section six we argue that the probability measure of five dimensional de Sitter gravity is given by the action of four dimensional conformal gravity. We end with a discussion.

2. Conformal scalar field

Before we discuss conformal gravity it is convenient to discuss the simpler case of a conformally coupled field with a fourth derivative action.

The reader is probably familiar with the fact that the action

$$S = \frac{1}{2} \int d^4x \sqrt{g} \left[(\nabla\phi)^2 + \frac{1}{6} \phi^2 R \right] \quad (2.1)$$

describes a conformally coupled field with dimension one. The action is invariant under $g \rightarrow \Omega^2 g$, $\phi \rightarrow \Omega^{-1} \phi$.

Here we are more interested in considering a conformally coupled field, C , of dimension zero. This is more similar to what we have for the metric, which also has dimension zero. The action is then of fourth order

$$S = \frac{1}{2} \int d^4x \sqrt{g} \left[(\nabla^2 C)^2 - 2(R_{\mu\nu} - \frac{1}{3}g_{\mu\nu}R) \partial_\mu C \partial_\nu C \right] \quad (2.2)$$

The curvature couplings are necessary for Weyl invariance ($g \rightarrow \Omega^2 g$, $C \rightarrow C$) [11], and are analogous to the usual one in (2.1).

Around flat space we simply have the fourth order equation $(\partial^2)^2 C = 0$ and the solutions are easy to find. If t is a time coordinate, then for a given spatial momentum \vec{k} the four solutions are $C = e^{\pm i|\vec{k}|t}$, $te^{\pm i|\vec{k}|t}$. As emphasized in [5], the Hamiltonian is not diagonalizable, and it has a Jordan form. This is due to t factor in the second solution. If we choose t as Euclidean time, we also have similar looking solutions but with $e^{\pm|k|t}$ instead.

We can now consider the same problem in AdS_4 with the metric

$$ds^2 = \frac{dz^2 + dx^2}{z^2} \quad (2.3)$$

This metric is equivalent, up to a Weyl transformation, to the flat space metric. Since this field is conformally coupled we expect that the answers are the same as the ones we would obtain in flat space. It is interesting, nevertheless, to consider the field action in AdS space. In that case (2.2) simplifies and gives

$$S = -\frac{1}{2} \int_{AdS_4} \sqrt{g} [(\nabla^2 C)^2 - 2(\nabla C)^2] \quad (2.4)$$

Introducing an extra field this can be rewritten as

$$S = \int_{AdS_4} \sqrt{g} \{[\nabla(C + \varphi)]^2 - [(\nabla\varphi)^2 - 2\varphi^2]\} \quad (2.5)$$

Integrating out φ we get back to (2.4), see [16]. The equation of motion for φ sets it equal to $\varphi = \frac{1}{2}\nabla^2 C$. Defining $\tilde{C} = C + \varphi$, we see that we have two scalar fields, one massless and the other with $m^2 = -2$, which is a tachyon in the allowed range. The fields have opposite kinetic terms. Thus one leads to states with positive norms and the other with negative norms. Which one produces positive norms and which one produces negative norms depends on the overall sign of the original action. We chose it in (2.4) so that the massless field gives rise to positive norm states. If we had done the same computation

in de-Sitter, then the sign of m^2 should be reversed. Note that the field C transforms in a single irreducible representation of the four dimensional conformal group $SO(2,4)$. On the other hand, if we consider the AdS problem, imposing suitable boundary conditions, we get two representations of $SO(2,3)$. For example, if we set the boundary conditions $C|_{z=0} = \partial_z C|_{z=0} = 0$, then we get highest weight representations with $\Delta = 3$ and $\Delta = 2$, one representation with positive norms and one with negative norms. One could imagine changing the sign of the norm for one of these $SO(2,3)$ representations by hand, but that would be in conflict with four dimensional conformal symmetry.

We note that even though the flat space Hamiltonian was not diagonalizable, the AdS global time Hamiltonian is diagonalizable and has a discrete spectrum. We have the unfortunate (but expected) feature that many states have negative norms. Now, if we view $z = 0$ as a boundary, it is very easy to understand why the AdS global time Hamiltonian is diagonalizable. This “time” corresponds to dilatations in the plane. Since the equation for C is conformal invariant we simply can consider it in flat space. Thus, dropping the denominator in (2.3) we get the flat space metric. The full $EAdS$ space corresponds to half of R^4 , with a boundary at $z = 0$. These flat space solutions can be expanded then in powers of z as $1, z^3; z, z^2$. The first two are the ones associated to the dimension $\Delta = 3$ operator or massless field. The second two and then the one corresponding to the dimension $\Delta = 2$ operator or tachyon field.

It is now clear that the operator with $\Delta = 2$ is sourced by the first derivative of C . In other words, the value of C at $z = 0$ is the source for the $\Delta = 3$ operator, and $\partial_z C$ at $z = 0$ is the source for the $\Delta = 2$ operator. In other words, in $EAdS$ we would set boundary conditions for $C = C_0(x)$ and $\partial_z C = C'_0(x)$. We can then find a solution which decays at $z \rightarrow -\infty$ and obeys these boundary conditions. Since the equations are fourth order it is clear that we can find such a solution. These two boundary conditions simply specify the boundary values of the two scalars we had in (2.5). If we consider a Euclidean AdS space and we compute the partition function setting $\partial_z C(z = 0) = 0$ but with nonzero boundary values of $C(z = 0)$, then we excite only one of the two quadratic AdS scalars, namely the massless one. Thus, in this way, only the field associated with positive norms is involved.

On the other hand, in Lorentzian AdS , the $\partial_z C = 0$ does not remove the ghosts. In fact we have normalizable states of the tachyon field which have negative norms. Since the tachyon field carries energy, it couples to the stress tensor. Then a gravitational wave

perturbation, or an insertion of the stress tensor, can create pairs of these negative norm states². We will not consider the Lorentzian AdS case any further.

Let us now consider the fourth order theory in flat Minkowski space. The wavefunctional of this theory can be written as a function of C and \dot{C} . For a quadratic theory these would be canonically conjugate variables. However, for a quartic theory these can be viewed as two “coordinate” variables. More precisely, we can say that the Poisson bracket between C and \dot{C} is zero. Thus, we can consider the wavefunctional $\Psi[C_B(x), \dot{C}_B(x)]$. We can evaluate it at $t = 0$. As usual, it can be computed either in the Lorentzian theory or in the Euclidean theory by the ordinary flat space analytic continuation. We can also view this as a computation of the wavefunctional in de-Sitter space, $ds^2 = \frac{-dt^2 + d\vec{x}^2}{t^2}$. This is equivalent because de Sitter is conformal to half of Minkowski space, $t \leq 0$. From the de-Sitter point of view, we compute the wavefunctional in the far future, on superhorizon scales, because we take the $t \rightarrow 0$ limit with x fixed.

This wavefunctional is given by evaluating the action on a solution of the equations of motion with appropriate boundary conditions. Rather explicitly, we can write this down in Fourier space

$$C(k, t) = C_B(k)(1 - ikt)e^{ikt} + C'_B(k)te^{ikt} \quad (2.6)$$

Inserting this into the action we get

$$\begin{aligned} iS &= -\frac{i}{2} \int d^3x \int_{-\infty}^0 dt [(\partial_t^2 - \partial_x^2)C]^2 = \frac{i}{2} \int d^3x [C\partial_t(\partial_t^2 - \partial_x^2)C - \partial_t C(\partial_t^2 - \partial_x^2)C]_{t=0} \\ iS &= \int \frac{d^3k}{(2\pi)^2} \left\{ -|C_B(\vec{k})|^2 k^3 + |C'_B(\vec{k})|^2 k + i2\text{Re}[C(\vec{k})C'_B(-\vec{k})]k^2 \right\} \end{aligned} \quad (2.7)$$

where in the first line we integrated by parts and used the equations of motion. We also used that for real profiles $C_B(x)$ we have $C_B(\vec{k}) = C_B(-\vec{k})^*$. These are very close to the Bunch Davies wavefunctions, $\Psi = e^{iS}$, for quadratic fields in de Sitter with $m^2 = 0$ and $m^2 = 2$, with positive and negative norms respectively. More precisely, if we ignore the purely imaginary term in the third line, then we get precisely the corresponding Bunch Davies wavefunctions. This imaginary term is a local term (involving $k^2 \sim \nabla^2$) which does not contribute to the expectation values. Thus if we set $C'_B = 0$ we recover the wavefunctional of an ordinary massless scalar field in de-Sitter.

² The pair actually has positive norm. However, if we separate them by a large amount in AdS space we will find that each one individually has negative norm.

Then the statement is simply that this wavefunctional

$$\Psi_{\text{Conformal}}[C_B(x), 0] = \Psi_{\text{BD-quad}}[C_B(x)] \quad (2.8)$$

where $\Psi_{\text{BD-quad}}$ is the usual Bunch-Davies wavefunction evaluated in the far future, on superhorizon distances³, for a massless scalar with a quadratic action. The dS to EAdS analytic continuation (see section 5 in [13]) becomes, in conformal gravity, the ordinary analytic continuation between lorentzian and Euclidean space. Notice that we are getting the scale invariant de Sitter wavefunction for C_B from a scale invariant action in four dimensions. The boundary condition $\partial_t C = 0$ is breaking the four dimensional conformal group, $SO(2, 4)$, to the three dimensional one, $SO(2, 3)$.

Finally, notice that in the de Sitter context one is often interested in computing expectation values of observables constructed from the scalar field C . These can be computed by considering the theory on R^4 with the additional condition $\partial_t C = 0$ at $t = 0$. This looks like a kind of brane at $t = 0$, with Neumann boundary conditions for the field C . Otherwise the field C can fluctuate in an arbitrary fashion, with vacuum boundary conditions in the future and the past. We can easily compute the propagator for the field C with these boundary conditions and use it to compute expectation values. From the quartic conformal scalar C , with the $\dot{C}(t = 0) = 0$ boundary condition, we get the expectation value

$$\langle C(t, \vec{k}) C(t', -\vec{k}) \rangle_{\text{conf}} \propto \frac{1}{k^3} \left[(1 - ikt)(1 + ikt') e^{ik(t-t')} - tt' k^2 (e^{ik(t-t')} - e^{ik(t+t')}) \right], \quad (2.9)$$

for $t \leq t' \leq 0$ (and a similar expression for $t' \leq t$). Here we have $k = |\vec{k}|$. We can compare this with the ordinary Bunch Davies expectation values for a massless scalar field

$$\langle \varphi(t, \vec{k}) \varphi(t', -\vec{k}) \rangle_{\text{BD-quadratic}} \propto \frac{1}{k^3} (1 - ikt)(1 + ikt') e^{ik(t-t')}, \quad t \leq t' \leq 0 \quad (2.10)$$

we see that it agrees with the first term in (2.9). In addition, they give identical results at $t = t' = 0$, which is the main statement in (2.8). Thus, the wavefunctions are not equal for all times, they are only identical when they are evaluated at $t = 0$. We can view the $t = 0$ slice where we impose the $\dot{C} = 0$ condition as a “Neumann S-brane”. The second term in (2.9) is simply due to the negative norm states which we are fixing at $t = 0$. Note that we can analytically continue these propagators to Euclidean time if necessary. It is a simple

³ We evaluate the wavefunctions at fixed comoving coordinate x as $t \rightarrow 0$.

matter to Fourier transform these propagators and express them in ordinary space. We encounter an IR divergence which is the usual IR divergence for a scalar field in de-Sitter space. From the quartic scalar we get an IR divergence also in flat space. Of course, here the two IR divergencies are identified with each other.

Note that the simplicity of the wavefunctions for a massless scalar in de-Sitter, contained in (2.10), is “explained” by the connection to a conformal scalar, but with a quartic action. Recall that for generic masses the fixed spatial momentum wavefunctions are given by Hankel functions.

Finally, note that expectation values can also be computed using a classical solution. If we consider a generating function for correlation functions $\langle e^{\int J(x)C(x)} \rangle$, then its expectation value can be obtained by considering the classical solution of the Euclidean equations of motion with the following boundary conditions

$$C(x) \rightarrow 0 \quad \text{as} \quad \tau \rightarrow \pm\infty ; \quad \partial_\tau C|_{\tau=0} = 0 , \quad \frac{i}{2} \partial_\tau^3 C|_{\tau=0} = J(x) \quad (2.11)$$

and C continuous across $\tau = 0$.

All that we have discussed here is for a free theory, in the sense that the action was quadratic in the field C . We will now turn to the gravity case, where we will be able to make similar statements, but for the full non-linear theory.

3. Conformal Gravity

We now turn to the case of gravity. The first observation is that the on shell action for four dimensional Einstein gravity in an Einstein space that is locally asymptotically $EAdS$ can be computed in terms of the action of Weyl gravity [10,17,18]. The argument is recalled in more detail in the appendix. Here let us just give a quicker version. We can write

$$\int W^2 = \int e + \int 2(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2) \quad (3.1)$$

where $e = R \wedge R^*$ is the Euler density, whose integral on a closed manifold gives a topological invariant. Now suppose that we have an Einstein space, with $R_{\mu\nu} = \alpha g_{\mu\nu}$ with α a constant. Evaluating the right hand side of (3.1) we get a constant times the volume from the Ricci tensors. Similarly the ordinary Einstein action $\int (R - 2\Lambda)$ gives us a constant times the volume. Thus the on shell action is proportional to the volume, and this volume, up to a topological term, is proportional to the Weyl action. The boundary terms that are

necessary to make the bulk integral of e into a proper topological invariant are the same as the counterterms that renormalize the volume [19,20,21,10,22,18], see appendix A for more details. The final statement is that

$$\int \sqrt{g}(R+6) - (\text{Counterterms}) = \frac{1}{4} \left[\int \sqrt{g} W^2 - E \right], \quad E = 32\pi^2 \chi \quad (3.2)$$

where χ is a topological invariant: the Euler number of the manifold with boundary, including the boundary terms. The “counterterms” subtract the infinite volume of the space near the boundary. They are unrelated to quantum mechanical counterterms of the bulk theory. All our computations are classical.

The second observation is that any space that is conformal to an Einstein space is a solution to the equations of motion of conformal gravity. The equations of motion of conformal gravity are $B_{\mu\nu} = 0$, where $B_{\mu\nu}$ is called the “Bach” tensor and its trace is zero because of the Weyl symmetry of the original action, see appendix C for its explicit form. Due to (3.1) the Bach tensor can be expressed in terms of derivatives of the Ricci tensor or Ricci scalar as well as some quadratic expression in the Ricci tensor or scalar. Let us evaluate $B_{\mu\nu}$ for an Einstein manifold, which obeys $R_{\mu\nu} = \alpha g_{\mu\nu}$, with α a constant. Then we find that R is a constant and that all the terms in $B_{\mu\nu}$ that contain derivatives vanish automatically. All terms that do not contain derivatives can only be proportional to $g_{\mu\nu}$, but since $B_{\mu\nu}$ is traceless we conclude that such a term should also automatically vanish.

Of course, the equations of motion of Weyl gravity also contain other solutions. Here we just point out that a simple boundary condition selects the solutions that are related to Einstein spaces.

This is argued by spelling out the form of the well known Starobinsky or Fefferman and Graham [23,24,25,20] expansion for a metric obeying the Einstein equations with a cosmological constant (an Einstein space) that is locally de-Sitter or Hyperbolic near the boundary

$$ds^2 = \frac{dz^2 + dx_i dx_j \left[g_{ij}^{(0)}(x) + z^2 g_{ij}^{(2)}(x) + z^3 g_{ij}^{(3)}(x) + \dots \right]}{z^2} \quad (3.3)$$

For de Sitter we have the same expansion with $z \rightarrow it$, up to an overall minus sign. Here we have performed the expansion in a particular gauge (basically a synchronous gauge) where we have set g_{zz} to a special value and g_{zi} to zero.

Note that in conformal gravity we can drop the $1/z^2$ overall factor and view this as the expansion of a manifold around $z = 0$. Then the one special property of (3.3) is that there is no linear term in z .

Thus, Einstein solutions are conformal to solutions which at $z = 0$ obey the condition $\partial_z g|_{z=0} = 0$. We call this a “Neumann” boundary condition.

Given that conformal gravity has fourth order equations, we expect that it has four solutions for a given spatial momentum. If we are in a EAdS, or dS situation, then we require that solutions either decay or have positive frequencies in the deep interior. This condition kills two of the solutions. The condition $\partial_z g_{ij} = 0$ kills another solution. So we are left with only one solution. Since an Einstein space is conformal to a solution obeying all the boundary conditions, we conclude that that is the solution that remains.

Actually the argument above was a bit too fast. Let us give a more accurate discussion that leads to the same conclusion. Conformal gravity has two physical modes with helicity two and one solution with helicity one (and the corresponding numbers with negative helicities)[26]. The $\dot{g}_{ij} = 0$ boundary condition sets to zero one of the helicity two modes, as well as the helicity one mode. This can be understood explicitly by writing down the general solution of the linearized Weyl equation (or Bach tensor). Writing the metric as a deformation of the flat space metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, imposing the gauge conditions $h_{00} = h_{i0} = h_{ii} = 0$ and linearizing the Bach tensor, one can write the general solution as⁴

$$h_{ij} \sim [(\epsilon_{ij} + k_{(i}\zeta_{j)}) - ik_0 t \hat{\epsilon}_{ij}] e^{ik \cdot x}, \quad k_\mu k^\mu = 0 \quad (3.4)$$

where ϵ_{ij} and $\hat{\epsilon}_{ij}$ are transverse ($k_i \epsilon_{ij} = 0$) and traceless ($\epsilon_{ii} = 0$) and describe the two spin two modes. ζ_i is also transverse ($k_i \zeta_i = 0$) and it describes the vector particle. We can easily see now that imposing $\dot{h}_{ij} = 0$ at $t = 0$, and a positive frequency condition in the past, we set the vector mode to zero and we get only one surviving spin two mode with $\epsilon_{ij} = \hat{\epsilon}_{ij}$ ⁵. This mode then agrees with the on shell graviton mode around de Sitter.

⁴ In this formula the indices ij run over three of the dimensions. The direction 0 could be time or the z direction, which is the direction orthogonal to the boundary.

⁵ As a side comment, one can view this surviving mode as a massless graviton and the vector together with the spin two mode as a “tachyonic” massive graviton. (It is tachyonic in the AdS case, in the dS case it is a massive graviton). Normally, a massive graviton also has a scalar component. However, for this very special value of the mass, we can have a particle with only the spin two and spin one parts. This is the phenomenon of “partial masslessness” described in [27]. If one added the Einstein action to the action of conformal gravity then the scale factor of the metric does not decouple any longer and we get a massless and a massive graviton with generic mass (and an extra scalar component). This type of setup was considered before with the idea of making the massive mode very massive. More recently this was also considered with the idea of

Note that the correspondence with conformal gravity “explains” the simplicity of these wavefunctions, which are simply plane waves up to the extra factor of time. Conformal gravity can also be formulated as a second order theory, in a way similar to what we saw for the scalar field in (2.5), see appendix C of [16]. In that formulation one can see more clearly the massless graviton mode, the vector and the tensor mode.

Note that the boundary condition $\partial_t g_{ij} = 0$ is not invariant under Weyl transformations. We can restrict the Weyl factor to obey $\partial_t \Omega = 0$ at $t = 0$. Alternatively, one could restate all the conditions in a more general gauge independent fashion. The resulting condition is to say that the $t = 0$ slice is a “totally geodesic” surface [29].

The conclusion is that classically, or at the level of tree diagrams, we have a complete equivalence between ordinary gravity and conformal gravity. It has been observed in [30], that conformal gravity around flat space suffers from a linearization instability. Namely, solutions of the linearized equations sometimes do not lift to solutions of the full non-linear equations. This is due to the presence of modes which are linear in time and the fact that the Hamiltonian cannot be diagonalized. These features are not present if we consider the problem on a half space, or equivalently the $EAdS$ or dS problems. For the case of the scalar field, we saw explicitly that one obtained two separated conformal towers under $SO(2,3)$. We expect that the same happens for the graviton. In fact, our arguments imply that conformal gravity (with the Neumann boundary condition) and ordinary gravity give the same answer for small but finite deformations around flat space. We have only checked explicitly that the boundary condition kills the wrong solutions at linearized order. However, we expect that the bulk differential operators have a spectrum with a gap, so that the boundary conditions continue to kill the wrong solutions in a small neighborhood of flat space. In particular, this is enough to establish the full tree level perturbative equivalence between conformal gravity with a Neumann boundary condition and ordinary gravity with a cosmological constant.

We conclude that at the level of tree diagrams we have the equality

$$\Psi_{\text{Conformal}}[g, \dot{g} = 0] \sim e^{c_W \int W^2} = \Psi_{\text{BD-ren}}(g) = e^{c_E \int \sqrt{g}(R \pm 6) - (\text{counterterms})} \quad (3.5)$$

where we have indicated that the equivalence is simply the statement that the classical actions evaluated on the corresponding classical solutions are the same. The \pm corresponds

making this mode degenerate with the graviton [28], which in effect removes the splitting that we have between the two spin two particles in AdS conformal gravity (see (2.4) for a discussion of a similar splitting between the modes of a scalar field).

to the $EAdS$ or dS cases. Note that conformal gravity has only a dimensionless coupling constant multiplying the whole action. Gravity in AdS or dS also has a dimensionless coupling set by $c \propto (M_{pl}R)^2$, with M_{pl} the (reduced) Planck mass⁶, and R the AdS or dS curvature radius. This is identified with the coupling constant appearing in conformal gravity. More precisely the relations are

$$\begin{aligned} c_W &= -i\frac{1}{8}(M_{pl}R_{dS})^2, & c_E &= i\frac{(M_{pl}R_{dS})^2}{2} = i\frac{M_{pl}^2}{2H^2}, & \text{for de Sitter} \\ c_W &= \frac{1}{8}(M_{pl}R_{EAdS})^2, & c_E &= \frac{(M_{pl}R_{EAdS})^2}{2}, & \text{for Euclidean AdS} \end{aligned} \quad (3.6)$$

These expressions could also be written in terms of the cosmological constant, which we have set here to $\Lambda = \pm \frac{3}{R^2}$. Note that in Euclidean Anti de Sitter we get the Weyl action with the “wrong” sign. Namely, the Weyl action has the nice feature that it is bounded below in Euclidean space. However, in the Euclidean Anti-de Sitter context we get it with the opposite sign. This is no problem in perturbation theory. Furthermore, this is physically reasonable since the two point function of the stress tensor, given by $\frac{\delta}{\delta g^{ij}(x)} \frac{\delta}{\delta g^{kl}(y)} \Psi[g]$, should be positive if the Euclidean gravity theory corresponds to the Euclidean continuation of a unitary boundary CFT (which is usually the case in AdS/CFT). On the other hand, the sign we get in the de-Sitter case is such that if we do the usual analytic continuation to Euclidean signature we get the “right” sign of the W^2 action. These facts are, of course, consistent with the observation in [13,15] that the de Sitter wavefunction can be obtained from the EAdS one by simply flipping the sign of $(M_{pl}R)^2$. This is just an overall sign in the exponent.

This connection might lead to a practical way to evaluate tree diagrams in AdS or dS , since in both cases we could view the computation as a computation around flat space in conformal gravity. The propagators and vertices of the two actions are different, but they both should give the same final answer. Given that twistor string theory contains conformal gravity [5], maybe one can also use that string theory to compute de Sitter correlators.

⁶ $M_{pl}^2 = \frac{1}{8\pi G_N}$.

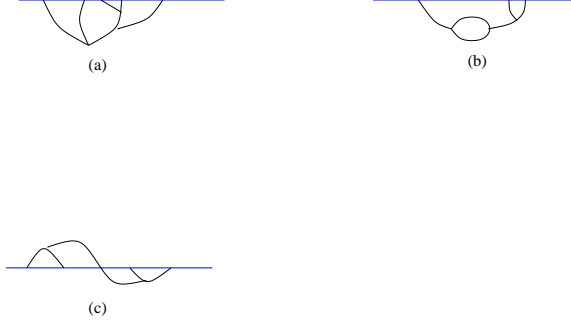


Fig. 1: (a) An example of a tree Feynman-Witten diagram that contributes to the tree level computation of the AdS partition function or the de Sitter wavefunction. We are stating that these diagrams give the same answer regardless of whether we compute them in Einstein gravity or in conformal gravity. (b) An example of a loop diagram that is *not* contained in the discussions of the present paper. (c) One example of a tree diagram that contributes to expectation values in de-Sitter. The top and bottom denote the two branches of the Schwinger-Keldysh contour. Alternatively, they can denote the upper or lower half space after analytic continuation to Euclidean signature.

In the case of dS computations, it is natural to integrate over the metric on the boundary and compute $\int \mathcal{D}g |\Psi(g)|^2$ ⁷. In conformal gravity this can be viewed as computing expectation values for the metric or the Weyl tensor in the presence of a Neumann S-brane, which is defined by setting $\partial_z g_{ij} = 0$. Namely, we set $\partial_z g_{ij}|_{z=0} = 0$ but we integrate over the value of g_{ij} at $z = 0$. At tree level this can be done, as in the case of the scalar field, by looking for classical solution of the Euclidean equations of motion with the boundary conditions that the space becomes flat as $\tau \rightarrow \pm\infty$ and that the metric is continuous at $\tau = 0$, and it obeys

$$\partial_\tau g_{ij}|_{\tau=0} = 0, \quad i\partial_\tau^3 g_{jl}|_{\tau=0} = J_{jl}(\vec{x}) \quad (3.7)$$

Then the classical action evaluated on this solution gives us $e^{-cS_{clas}} = \langle e^{\int J_{ij} h^{jl}} \rangle$, here h is a deformation from flat space, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

4. Black holes

The arguments we presented above establish the equivalence for small (but finite) deformations around flat space (or an S^3) on the boundary. We could wonder if there are

⁷ Of course, this integral should be defined carefully. In particular, we do not integrate over the overall scale factor in the metric. In conformal gravity, this is clear, since that is gauge symmetry. In the de Sitter case, this can be viewed as selecting a “time” slice. Also, at tree level we just use saddle point and the details of the measure do not matter.

new solutions once we consider large enough deformations. As an initial exploration, we consider a boundary of the form $S^1 \times S^2$, so that we get contributions from Euclidean black holes. Black hole solutions in conformal gravity were considered in [31,32,33]. We can write down the general spherically symmetric ansatz for conformal gravity as

$$ds^2 = e^{2g} dt^2 + \frac{d\rho^2}{e^{2g}} + d\Omega_2^2 \quad (4.1)$$

where $d\Omega_2$ is the metric of the two sphere. We have used the fact that, in conformal gravity, we can perform a Weyl transformation that sets the radius of the two sphere to one. The equations of motion of conformal gravity imply

$$-(g'')^2 + 4(g')^4 + 2g^{(3)}g' + 8(g')^2g'' + e^{-4g} = 0 \quad (4.2)$$

Note that the equation is cubic and not quartic. This reduction in order also arises in Einstein gravity and it is due to the reparametrization symmetry constraint⁸. We set the boundary at $\rho = 0$, and set $g'(0) = 0$ at this point. We can also use a symmetry under rescalings of the coordinates to set $g(\rho = 0) = 0$. Since (4.2) is a cubic equation we expect a three parameter family of solutions. We have already fixed two of the parameters by using symmetries of the equation. Thus the solution depends only on one non-trivial parameter. The general solution is

$$e^{2g} = (1 \pm \rho^2 - 2m\rho^3) \quad (4.3)$$

There are two branches of solutions, associated to the \pm signs. Let us discuss first the branch with the plus sign, or $g''(0) = 1$. This branch describes the ordinary *AdS* black holes, with $m > 0$. Of course, $m = 0$ describes global *AdS*. In other words, up to an overall factor of r^2 , the metric (4.1) is the same as [34]

$$ds^2 = (r^2 + 1 - 2m/r)dt^2 + \frac{dr^2}{(r^2 + 1 - 2m/r)} + r^2 d\Omega_2^2 \quad (4.4)$$

together with the identification $\rho = 1/r$. The free energy of the black hole can be computed using conformal gravity. We get

$$\frac{M_{pl}^2 R_{AdS}^2}{8} \int W^2 = 8\pi^2 M_{pl}^2 R_{AdS}^2 \left[\frac{(1 + \rho_h^2)^2}{\rho_h^2(3 + \rho_h^2)} \right] ; \quad \beta = \frac{4\pi\rho_h}{\rho_h^2 + 3} \quad (4.5)$$

⁸ In other words, if we introduce N via $d\rho^2 \rightarrow Nd\rho^2$ in (4.1), then (4.2) is the equation of motion for N , at $N = 1$.

where ρ_h is a root of $e^{2g} = 0$. In particular, the entropy of the solution can also be computed using Wald's formula [35]. The Wald entropy of a black hole in a theory where the lagrangian depends on the curvature, $L(g, R_{\mu\nu\rho\sigma})$, is given by [36]

$$S = -2\pi \int_{\Sigma_2} L^{\mu\nu\rho\sigma} \epsilon_{\mu\nu} \epsilon_{\rho\sigma} , \quad L^{\mu\nu\rho\sigma} = \frac{\partial L}{\partial R_{\mu\nu\rho\sigma}} \propto W^{\mu\nu\rho\sigma} \quad (4.6)$$

where Σ_2 is the horizon and $\epsilon_{\mu\nu}$ is the binormal, normalized so that $\epsilon_{\mu\nu} \epsilon^{\mu\nu} = -2$. For the above solution this simply gives

$$S = \frac{4\pi R_{AdS}^2 (r_h^2 + 1)}{4G_N} \quad (4.7)$$

where r_h is the horizon radius. The constant factor is due to the contribution from the Euler character. Since the black hole has a different topology, the Euler character in (3.2) gives us a non-trivial contribution. (The Euler character of the black hole is $\chi = 2$). This extra contribution to the partition function (or βF) is independent of the temperature and thus, it gives a constant contribution to the entropy⁹. As usual, these Schwarzschild AdS black holes exist for inverse temperatures bigger than $\beta \leq \frac{2\pi}{\sqrt{3}}$.

One can ask the following question. Imagine that we evaluate the Wald entropy formula (4.6) for conformal gravity, on a solution of Einstein's gravity. Then can we show that it is equal to the area?. In fact, it is easy to show that this is the case. Writing $\int W^2 - E$ as in (3.1), the Wald's formula gives us terms involving the Ricci tensor. If we have an Einstein space these become simply the metric. Thus, it is always automatic that Wald's formula on conformal gravity (minus the Euler character) on an Einstein space reproduces the results of Einstein gravity. Of course, this is also a consequence of the equivalence between the two on shell actions.

⁹ As a completely side remark, notice that if we have Einstein gravity plus a correction proportional to the Euler character, then the fact that the black hole entropy should be positive sets a bound on the size of the coefficient of the Euler character term. This puts a bound for one sign of the coefficient. For the other sign of the coefficient, we can set a bound by demanding that the entropy of the black hole is less than that of the Hawking radiation once it has evaporated completely (this bound depends on the number of light species). The bound is set by the lowest distance scale at which we trust the black hole solutions. The bound on the Euler coefficient is then of the order of $(4\pi)^2 \frac{M_{pl}^2}{M_s^2}$ where M_s is the energy scale at which we cease to trust Einstein gravity.

In addition to the ordinary black hole solutions, the equations of conformal gravity have a second branch of solutions if we choose the minus sign in (4.3), or $g''(0) = -1$. These are additional solutions which exist with our boundary conditions. There is a single solution for each value of the temperature. The “origin” of these solutions can be understood as follows. In Einstein gravity, we have a related family of black hole solutions obtained by replacing S^2 by H^2 and $(r^2 + 1 - 2m/r) \rightarrow (r^2 - 1 - 2m/r)$ in (4.4) [37], with $m \geq -\frac{1}{\sqrt{27}}$. In conformal gravity these are also solutions of the S^2 problem for the following reason. Starting from H^2 we can make an analytic continuation which takes $ds_{H^2}^2 \rightarrow -ds_{S^2}^2$. We can also take $e^{2g} \rightarrow -e^{2g}$, which is a symmetry of (4.2). Finally we can make a Weyl transformation by an overall minus sign in the metric. This chain of arguments explains “why” we found another solution for the S^2 problem. Of course, it is a simple matter to check that (4.3) is a solution of (4.2). The two branches are distinguished by the value of $g''(0)$ and we can imagine selecting the correct branch by selecting the appropriate sign. In fact, in an Einstein space, the Fefferman Graham expansion (3.3) fixes the value of the second derivative of the metric around the boundary. However, from the point of view of conformal gravity this seems unnatural.

5. The Hartle Hawking factor and the sphere partition functions

As a side remark, notice that the de Sitter Hartle Hawking factor can also be obtained in all dimensions from the analytic continuation from Euclidean AdS, see also [15]. In the case of even bulk dimensions, this Hartle Hawking factor is the same (more precisely it is the square) of the sphere partition function of the boundary field theory (if that theory were to exist). Namely, it is the de Sitter analog of the sphere partition functions that have been recently computed in [38,39,40], and references therein. For example, consider the case of four bulk dimensions, and three boundary dimensions. If we have an S^3 boundary the field theory partition function for a theory that has an AdS_4 dual is computed by

$$\log Z = \frac{(M_{pl} R_{AdS})^2}{2} v_{S^3}(-6) \int_0^{\rho_c} d\rho (\sinh \rho)^3; \quad (5.1)$$

$$\int_0^{\rho_c} d\rho (\sinh \rho)^3 = \frac{e^{3\rho_c}}{24} - \frac{3e^{\rho_c}}{8} + \frac{2}{3} + o(e^{-\rho_c})$$

The sphere partition function is obtained by taking the $\rho_c \rightarrow \infty$ limit, discarding the divergent terms. Thus, it is given by the factor of $2/3$ in the square brackets. Here

$v_{S^3} = 2\pi^2$ is the volume of a three sphere. On the other hand, we can compute the Hartle Hawking factor

$$\log |\Psi|_{HH}^2 = \frac{(M_{pl} R_{dS})^2}{2} v_{S^3} 6 \left[2 \int_0^{\frac{\pi}{2}} d\rho (\sin \rho)^3 \right] \quad (5.2)$$

$$\int_0^{\frac{\pi}{2}} d\rho (\sin \rho)^3 = \frac{2}{3}$$

These two factors are equal (up to a factor of two, since we need to multiply (5.1) by $|\Psi|^2$ in (5.2)). The equality holds, after we analytically continue $R_{AdS}^2 \rightarrow -R_{dS}^2$. We can formally go from (5.1) to (5.2) by taking $\rho \rightarrow \rho + i\pi/2$ and $R_{AdS} \rightarrow -iR_{dS}$. See a more detailed discussion in [15].

Thus, from a dS/CFT perspective, the Hartle Hawking factor is the S^3 partition function of the dual field theory, with the local infinities subtracted.

Of course, if one were to replace the S^3 by other manifolds, such as a $S^1 \times S^2$, or T^3 , then one would get different answers. In fact, black branes in AdS_4 can be viewed as computing such factors, due to the analytic continuation from $EAdS$ to dS .

This works in a similar way in other even bulk dimensions. For odd bulk dimensions, there is a logarithmic divergence and the Hartle Hawking factor is related to the coefficient of this divergence, as we will see explicitly for the five dimensional case in the next section.

6. Four dimensional conformal gravity and five dimensional Einstein gravity

In this section we discuss a different appearance of four dimensional conformal gravity from de Sitter space. We will show that 4 dimensional Euclidean conformal gravity is the late time (superhorizon) measure factor arising from tree level five dimensional gravity in de-Sitter space.

Before considering the de Sitter problem, let us review a well known property of the Anti-de-Sitter case. Let us consider evaluating the action for an asymptotically hyperbolic space which is a Euclidean solution of five dimensional Einstein gravity with a cosmological constant. We fix a boundary metric of the form $\frac{\hat{g}_{ij}}{\epsilon^2}$. We also have a bulk Einstein metric, obeying the usual five dimensional Einstein manifold condition, $R_{\mu\nu} = -4g_{\mu\nu}$ with a boundary condition set by the four dimensional metric \hat{g} .

Then the on shell Euclidean action ($Z \sim e^{-S}$) has the following expansion [12]

$$\begin{aligned} -S(\hat{g}) &= \frac{M_{pl}^3 R_{AdS}^3}{2} \left[\int d^5x \sqrt{g}(R + 12) + 2 \int d^4x K \right] = \\ &= \frac{M_{pl}^3 R_{AdS}^3}{2} \left[\frac{a_0}{\epsilon^4} \int d^4x \sqrt{\hat{g}} + \frac{a_2}{\epsilon^2} \int d^4x \sqrt{\hat{g}} \hat{R} + \frac{\log \epsilon}{8} \int \sqrt{\hat{g}}(\hat{W}^2 - \hat{e}) \right] - S_R[\hat{g}] \end{aligned} \quad (6.1)$$

where S_R is finite as $\epsilon \rightarrow 0$ and a_0, a_2 are two (real) numerical constants¹⁰.

We recognize the action of conformal gravity in the coefficient of the logarithmic term [6]¹¹.

Now, let us consider the case of pure de-Sitter gravity in five dimensions. In that case, we can similarly evaluate the wavefunction, as we did in four dimensions. This wavefunction can be evaluated from analytic continuation from the Euclidean AdS case we mentioned above. Namely, we set

$$z = -i\eta, \quad R_{AdS} = -iR_{dS} \quad (6.2)$$

This implies that the cutoff $\epsilon_z = -i\epsilon_\eta$. In de Sitter, $\eta, \epsilon_\eta < 0$. As in the four dimensional case, all divergent “counterterms” become imaginary, so that they drop out from the quantum measure, $|\Psi|^2$. On the other hand, in five dimensions, the finite part, given by S_R , *also* becomes purely imaginary and drops out. The only real term comes from the analytic continuation of the logarithmic term $\log \epsilon_z \rightarrow \log(-\epsilon_\eta) + i\pi/2$. This gives a square of the wavefunction of the form

$$|\Psi[\hat{g}]|_{\text{Einstein}}^2 = \exp \left[-\frac{M_{pl}^3 R_{dS}^3}{2} \frac{\pi}{8} \int d^4x \sqrt{g}(\hat{W}^2 - \hat{e}) \right] \quad (6.3)$$

One can check explicitly that this gives the right value for the quadratic fluctuations [13]. Here we claim that this captures all the tree level diagrams in de-Sitter space. One remarkable feature of this action is that it is purely local in space. This is in contrast to the four dimensional case, where $|\Psi|^2$ has a non-local expression. This is, of course, just a tree level result. As we include loops we could generate non-local terms in the effective action. Loop diagrams would not have the factor of $(MR)^3$ which will make their analytic continuation to de-Sitter different.

¹⁰ $a_0 = 6$. In d boundary dimensions, this first coefficient is $a_0 = 2(d-1)$, which is positive.

¹¹ If we had maximally supersymmetric 5d gauge supergravity in the bulk, then we would get the action of $\mathcal{N} = 4$ conformal supergravity as the coefficient of the logarithmic term [6].

Now, if we want to compute expectation values of the metric in five dimensional de-Sitter space, then we could do it in the following way. First we compute $|\Psi[\hat{g}]|^2$, which gives the action of four dimensional conformal gravity. Then we can do the functional integration over boundary metrics. This then becomes a problem in four dimensional conformal gravity. Again, if we restrict to tree level results, we have a precise equivalence with tree level computations in four dimensional conformal gravity. Note that in the five dimensional de-Sitter problem we can compute probabilities by slicing the geometry at various times. In terms of the wavefunction of the universe, this translates into various scale factors for the geometry. The measure $|\Psi|^2 \sim e^{-S_{\text{conf}}}$ is explicitly independent under the choice of scale factor for the spatial metric due to the exact Weyl invariance of the action of four dimensional gravity. Furthermore, when we compute expectation values with this measure we *do not* integrate over the scale factor. From the point of view of the de-Sitter, we do not integrate because probabilities from the Wheeler de Witt wavefunction are computed at a given time (or scale factor). From the point of view of conformal gravity, the invariance under rescalings of the scale factor is a gauge symmetry, thus, we “divide by the volume of the gauge group”, which amounts to fixing a particular scale factor.

In summary, *late times, superhorizon, expectation values of the metric in five dimensional de-Sitter space are equal, at tree level, to expectation values computed using four dimensional conformal gravity*. In other words, the tree level cosmological superhorizon measure in five dimensional gravity with a cosmological constant is given by four dimensional conformal gravity.

As a simple check of these formulas, let us compute the five dimensional Hartle Hawking factor using (6.3). The Hartle-Hawking factor is a real term that can be interpreted as giving us the probability of making a universe with S^4 spatial topology [41]. (Recall that we are in five dimensions, so S^4 is the spatial slice.) It is given by evaluating the five dimensional Einstein action on an S^5 (and it equals the entropy of dS_5). It is

$$|\Psi|^2 \sim \exp \left[\frac{M_{pl}^3 R_{dS}^3}{2} \int_{S^5} \sqrt{g}(R - 12) \right] = \exp \left[\frac{M_{pl}^3 R_{dS}^3}{2} 8\pi^3 \right] \quad (6.4)$$

where π^3 is the volume of the five sphere.

We get the same from (6.3). For S^4 the Weyl tensor vanishes because S^4 is conformally flat. Then only the piece involving the Euler number survives, where $\chi(S^4) = 2$. Thus we get

$$|\Psi|^2 \sim \exp \left[\frac{M_{pl}^3 R_{dS}^3}{2} \pi 4\pi^2 \chi(S^4) \right] = \exp \left[\frac{M_{pl}^3 R_{dS}^3}{2} 8\pi^3 \right] \quad (6.5)$$

7. Discussion

We have shown that four dimensional conformal gravity with a Neumann boundary condition is classically equivalent to ordinary four dimensional Einstein gravity with a cosmological constant. This equivalence was shown only for small but finite deformations around a flat (or S^3 boundaries). These two theories are equivalent for the computation of the “renormalized” wavefunction, or partition function. In the $EAdS$ context conformal gravity computes the renormalized partition function. In the dS case, it computes the superhorizon part of the wavefunction. Alternatively, we can say that it computes the tree diagrams for gravity wave fluctuations outside the horizon, see fig. 1. This relation might be useful for computing tree diagrams in de Sitter or AdS. For large deformations of the geometry we found new solutions, though it might be the case that one can get rid of these extra solutions. We considered boundaries with $S^1 \times S^2$ geometry which give rise to bulk configurations corresponding to Euclidean black holes.

It would be more exciting if we could also use conformal gravity to compute such wavefunctions at the quantum level. The reason is that a $\mathcal{N} = 4$ supersymmetric version is believed to be finite¹². One could compute the wavefunctions at one loop in conformal gravity with the Neumann boundary condition. These are interpreted as superhorizon wavefunctions and it is not clear how to check that they correspond to wavefunctions in a unitary theory. Maybe the quantum corrected wavefunction is not Weyl invariant.

Another interesting problem is to generalize the Neumann boundary condition to the full $\mathcal{N} = 4$ conformal gravity case. Here one would expect to put boundary conditions that remove some of the fields of the conformal gravity theory. A quick look at Table 1 of [5], shows that if we remove all fields which do not come in doublets, and we retain one field per doublet, then we get the physical field content of $SO(4)$ gauged supergravity in four dimensions¹³. However, we have not found the full non-linear embedding of a solution of $SO(4)$ gauged supergravity into $\mathcal{N} = 4$ conformal supergravity. It would be interesting whether this work (or some variation of it).

¹² Loop corrections in pure bosonic conformal gravity generate bulk logarithmic divergencies. While in quantum field theory such divergencies are not a problem (if the theory is asymptotically free), in the context of conformal gravity, where the Weyl symmetry is a gauge symmetry, we should discard these theories as anomalous [42]. These divergencies cancel in some specific versions of $\mathcal{N} = 4$ conformal gravity, one of which involves coupling it to a rank four Super Yang Mills theory [43,11,44,5,11].

¹³ This was suggested by N. Berkovits.

$\mathcal{N} = 4$ conformal gravity also includes a scalar field with dimension zero, like the one we discussed in section two. In conformal gravity it is possible to couple the scalar field to the Weyl tensor as $\int f(C)W^2 + \dots$. This function cannot be removed by a Weyl transformation, as in Einstein gravity. More precisely, we have a complex field C such that the action has the form $\int f(C)W_+^2 + f(C)^*W_-^2 + \dots$. The conformal gravity theory that arises from the twistor string theory has the form $S = \int e^C W_+^2 + \dots$. Now, as we mentioned around (3.6) this overall factor in the action is the ratio of the Planck scale to the de-Sitter radius. Thus, in such a theory, one could get a very large ratio between these two scales with a moderate value of C . The theory is reminiscent of Liouville theory in two dimensions.

We have also mentioned the fact that the de Sitter Hartle Hawking factor, related to the de-Sitter entropy, $e^{-S_{dS}}$ can be viewed as the S^3 partition function of a hypothetical dual field theory that lives at the S^3 boundary of dS_4 . This is a side remark, independent of the relation with conformal gravity.

This could be generalized to higher even dimensions. In six dimensions, we need two boundary conditions on the metric, on the first derivative and on the second derivative. In that case the bulk action is sixth order and it has the rough form $\int W \nabla^2 W$ (the precise form is given in eqn. (3.6) of [17]).

It would be interesting to understand whether the flat space tree level S-matrix of pure gravity could also be computed in terms of a computation in conformal gravity. It seems that if we compute amplitudes in conformal gravity which involve only the spin two solutions that do not increase in time, then we could select the Ricci flat solutions. The amplitude should then follow from evaluating a suitable boundary term, since the on shell bulk action would be zero.

Going now to five dimensions, we have shown that the square of the wavefunction of the universe of five dimensional pure gravity is given by the action of conformal gravity. This follows in a simple way from the well known expansion of the wavefunction at late times and the connection (via analytic continuation) to the Euclidean AdS_5 problem. It seems surprising that the wavefunction is completely local, in stark contrast with the one in dS_4 .

Acknowledgments

I am specially grateful to Alice Chang for bringing [10,17] to my attention. I also thank N. Arkani Hamed, N. Berkovits, S. Deser, D. Harlow, D. Hofman, A. Strominger, A. Tseytlin and E. Witten for discussions. This work was supported in part by U.S. Department of Energy grant #DE-FG02-90ER40542.

Appendix A. Renormalized Einstein action as the conformal gravity action

Anderson showed that renormalized volumes in asymptotically hyperbolic Einstein spaces are given by the Weyl gravity action [10]. Here we simply outline his argument.

We are interested in computing the renormalized Einstein action, given by [19,20]

$$-S_{E,ren} = \lim_{\epsilon \rightarrow 0} \left[\int_{\Sigma_{4,\epsilon}} (R + 6) + 2K - 4 \int_{\Sigma_3} \sqrt{g} - \int_{\Sigma_3} \sqrt{g} R^{(3)} \right] \quad (\text{A.1})$$

where we have set the *AdS* radius to one. For an Einstein manifold we have that $R_{\mu\nu} = -3g_{\mu\nu}$, so that evaluating the Einstein action is the same (up to a coefficient) to computing the volume. More precisely, we get

$$-S_{E,ren} = -6V_{ren} = -6 \lim_{\epsilon \rightarrow 0} \left[\int_{\Sigma_4} \sqrt{g} - \frac{1}{3} \int_{\Sigma_3} \sqrt{g} - \frac{1}{2} \int_{\Sigma_3} \sqrt{g} R^{(3)} \right] \quad (\text{A.2})$$

In order to evaluate this [10] starts from the expression for the Euler density $e = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2$, so that $E = \int \sqrt{g}e$ is a topological invariant on a closed manifold. We then use the fact that

$$W^2 - e = 2(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2) \quad \rightarrow \quad -6.4 \quad (\text{A.3})$$

where we have written the form of the right hand side for an Einstein space. Thus we see that integrating the left hand side we get, up to a number, the volume of the space. At this point we might be confused because the right hand side is infinite, while the integral of W^2 would be finite. What happens is that the integral of e is not a topological invariant unless we add some boundary terms. These boundary terms have a simple Chern Simons form, so that the full topological invariant is

$$E = \int_M \epsilon_{abcd} R^{ab} \wedge R^{cd} - \int_{\partial M} \epsilon^{abcd} [\omega^{ab} d\omega^{cd} + \frac{2}{3} \omega^{ab} (\omega^{cf} \omega^{fd})] \quad (\text{A.4})$$

where w is the spin connection. This differs from what we had in (A.3) by this last term. This last term is actually divergent as we approach the *AdS* boundary. In fact, for an Einstein space, it gives purely divergent terms which are the counterterms that renormalize the Einstein term and no finite residual term. This is expected since the divergent terms have to cancel out the divergences properly and there is no local invariant expression of the boundary metric that could remain as a possible finite term. Thus in conclusion we find that

$$\frac{1}{4} \left[\int W^2 - E \right] = S_{Einstein-ren} , \quad \chi = \frac{1}{32\pi^2} E \quad (\text{A.5})$$

Note that the left hand side is now fully conformal invariant and it can be computed with any metric in the same conformal class. In particular, by dropping the $1/z^2$ factor in (3.3) we get a metric which is a small deviation around flat space. Moreover, if we consider small deviations from AdS , then we find that $E = 0$ (since it is zero for flat space and it is a topological invariant). The conventionally normalized Euler character is χ in (A.5).

This argument has been extended to other dimensions in [17].

Appendix B. Norms of states in conformal gravity

In a conformal field theory it is useful to consider the theory on $S^3 \times R$ and to study the states of the theory on such a space. We can do the same for conformal gravity and we can classify the quadratic fluctuations using the conformal symmetry. In other words, we fix the background metric to $S^3 \times R$ and we study the linearized theory, with a linearized gauge symmetry. In the full theory, the conformal symmetries are constraints on the quantum state of the theory, so that a full non-linear analysis is more complicated and it will involve some evolution on the background metric. We ignore this in our discussion. We also use the usual state-operator mapping to discuss the states of the theory.

We simply would like to point out that all the states of a graviton are in a single representation of the conformal group. Via the state operator correspondence, we find that all states are in a single representation of the conformal group. They are all descendents of the Weyl tensor W . More precisely, we have two representations, one coming from W^+ and one from W^- .

It is interesting to compute the norms of these states. All norms can be computed by the use of the conformal algebra commutation relations, $[K, P] \sim D + M$ and $P^\dagger = K$ relation. All norms are then computed in terms of the norm of the lowest weight state. Since this state is simply the Weyl operator, it has conformal dimension two and spin two. $(\Delta, S) = (2, 2)$ lies outside the unitarity bound and for this reason its descendents have both positive norm and negative norm states. This computation also shows that there is a null state (orthogonal to all states) which is simply zero by the (linearized) equations of motion for the Weyl tensor.

It is also interesting to consider the scalar field C . Naively we should start from a conformal dimension zero primary. However, as in two dimensions, it is better to view $\partial_\mu C(0)$ as the primary. Its descendents then have both positive and negative norms. There is a null state, which is simply the equation of motion for C . The field C itself is not a good conformal field, due to IR divergencies. Exponentials such as $e^{\alpha C}$ are good conformal operators. The field C is similar to the scalar field of Liouville theory in two dimensions.

Appendix C. Formulas for the Bach tensor

Here we write a few useful formulas. We follow Wald's conventions [45]. The following divergence of the Weyl tensor is

$$\nabla_\alpha W^\alpha_{\mu\nu\rho} = -\frac{1}{2} \left[\nabla_\rho R_{\mu\nu} - \nabla_\nu R_{\mu\rho} + \frac{1}{6} (g_{\mu\rho} \nabla_\nu R - g_{\mu\nu} \nabla_\rho R) \right] = C_{\mu\nu\rho} \quad (\text{C.1})$$

We have that $\nabla^\mu C_{\mu\nu\rho} = 0$, $C_{[\mu\nu\rho]} = 0$, $C_{\mu\nu\rho}$ is called the Cotton tensor. The equations of motion of conformal gravity involve the Bach tensor [31]

$$\begin{aligned} B_{\mu\nu} &= 2\nabla^\beta \nabla^\alpha W_{\alpha\mu\nu\beta} + W_{\alpha\mu\nu\beta} R^{\alpha\beta} = 2\nabla^\alpha \nabla^\beta W_{\alpha\mu\nu\beta} + W_{\alpha\mu\nu\beta} R^{\alpha\beta} \\ B_{\mu\nu} &= \nabla_\alpha \nabla_\mu R^\alpha_\nu + \nabla_\alpha \nabla_\nu R^\alpha_\mu - \nabla^2 R_{\mu\nu} - \frac{2}{3} \nabla_\mu \nabla_\nu R + \frac{1}{6} g_{\mu\nu} \nabla^2 R \\ &\quad - 2R_{\alpha\mu} R^\alpha_\nu + \frac{2}{3} R^{\mu\nu} R + \frac{1}{2} g_{\mu\nu} (R_{\alpha\beta} R^{\alpha\beta} - \frac{1}{3} R^2) \end{aligned} \quad (\text{C.2})$$

References

- [1] S. L. Adler, Rev. Mod. Phys. **54**, 729 (1982).
- [2] P. D. Mannheim, [arXiv:1101.2186 [hep-th]].
- [3] G. 't Hooft, arXiv:1104.4543 [gr-qc].
- [4] E. Bergshoeff, M. de Roo, B. de Wit, Nucl. Phys. **B182**, 173 (1981). B. de Wit, J. W. van Holten, A. Van Proeyen, Nucl. Phys. **B184**, 77 (1981).
- [5] N. Berkovits, E. Witten, JHEP **0408**, 009 (2004). [hep-th/0406051].
- [6] H. Liu, A. A. Tseytlin, Nucl. Phys. **B533**, 88-108 (1998). [hep-th/9804083].
- [7] V. Balasubramanian, E. G. Gimon, D. Minic, J. Rahmfeld, Phys. Rev. **D63**, 104009 (2001). [hep-th/0007211].
- [8] C. M. Bender, P. D. Mannheim, Phys. Rev. Lett. **100**, 110402 (2008). [arXiv:0706.0207 [hep-th]].
- [9] A. V. Smilga, SIGMA **5**, 017 (2009). [arXiv:0808.0139 [quant-ph]].
- [10] M. Anderson, Math. Res. Lett., 8 (2001) 171-188, arXiv:math/0011051
- [11] E. S. Fradkin, A. A. Tseytlin, Phys. Rept. **119**, 233-362 (1985).
- [12] M. Henningson, K. Skenderis, JHEP **9807**, 023 (1998). [hep-th/9806087].
- [13] J. M. Maldacena, JHEP **0305**, 013 (2003). [astro-ph/0210603].
- [14] P. McFadden, K. Skenderis, Phys. Rev. **D81**, 021301 (2010). [arXiv:0907.5542 [hep-th]].
- [15] D. Harlow, D. Stanford, [arXiv:1104.2621 [hep-th]].
- [16] R. R. Metsaev, arXiv:0707.4437 [hep-th].
- [17] A. Chang, J. Qing and P. Yang, arXiv:math/0512376
- [18] O. Miskovic, R. Olea, Phys. Rev. **D79**, 124020 (2009). [arXiv:0902.2082 [hep-th]].
- [19] V. Balasubramanian and P. Kraus, Commun. Math. Phys. **208**, 413 (1999) [arXiv:hep-th/9902121].
- [20] K. Skenderis, Class. Quant. Grav. **19**, 5849 (2002) [arXiv:hep-th/0209067].
- [21] R. Aros, M. Contreras, R. Olea, R. Troncoso, J. Zanelli, Phys. Rev. Lett. **84**, 1647-1650 (2000). [gr-qc/9909015].
- [22] R. Olea, JHEP **0506**, 023 (2005). [hep-th/0504233].
- [23] A. A. Starobinsky, JETP Lett. **37**, 66 (1983).
- [24] C. Fefferman and C.R. Graham, Conformal Invariants, in Elie Cartan et les Mathematiques daujourdhui (Asterisque, 1985) 95.
- [25] S. de Haro, S. N. Solodukhin and K. Skenderis, Commun. Math. Phys. **217**, 595 (2001) [arXiv:hep-th/0002230].
- [26] R. J. Riegert, Phys. Lett. **A105**, 110-112 (1984).
- [27] S. Deser, R. I. Nepomechie, Annals Phys. **154**, 396 (1984). S. Deser and A. Waldron, Phys. Rev. Lett. **87**, 031601 (2001) [arXiv:hep-th/0102166].

- [28] H. Lu, C. N. Pope, Phys. Rev. Lett. **106**, 181302 (2011). [arXiv:1101.1971 [hep-th]].
S. Deser, H. Liu, H. Lu, C. N. Pope, T. C. Sisman, B. Tekin, Phys. Rev. **D83**, 061502 (2011). [arXiv:1101.4009 [hep-th]].
- [29] A. Chang, S-Y. S. Chen, P. Yang, Private Communication.
- [30] D. G. Boulware, G. T. Horowitz, A. Strominger, Phys. Rev. Lett. **50**, 1726 (1983).
- [31] R. Bach, Math. Zeit., **9** , 110 (1921).
- [32] R. J. Riegert, Phys. Rev. Lett. **53**, 315-318 (1984).
- [33] S. Deser, B. Tekin, Class. Quant. Grav. **20**, 4877-4884 (2003). [gr-qc/0306114].
- [34] S. W. Hawking, D. N. Page, Commun. Math. Phys. **87**, 577 (1983).
- [35] R. M. Wald, Phys. Rev. D **48**, 3427 (1993) [arXiv:gr-qc/9307038].
- [36] T. Jacobson, G. Kang, R. C. Myers, Phys. Rev. **D49**, 6587-6598 (1994). [gr-qc/9312023].
- [37] R. Emparan, JHEP **9906**, 036 (1999). [hep-th/9906040].
- [38] A. Kapustin, B. Willett, I. Yaakov, JHEP **1010**, 013 (2010). [arXiv:1003.5694 [hep-th]].
- [39] N. Drukker, M. Marino, P. Putrov, [arXiv:1007.3837 [hep-th]].
- [40] D. L. Jafferis, [arXiv:1012.3210 [hep-th]].
- [41] J. B. Hartle, S. W. Hawking, Phys. Rev. **D28**, 2960-2975 (1983).
- [42] A. Strominger, V. P. Nair, Phys. Rev. **D30**, 2528 (1984).
- [43] E. S. Fradkin, A. A. Tseytlin, Phys. Lett. **B134**, 187 (1984).
- [44] H. Romer, P. van Nieuwenhuizen, Phys. Lett. **B162**, 290 (1985).
- [45] R. M. Wald, "General Relativity," Chicago, Usa: Univ. Pr. (1984) 491p.